JOURNAL OF APPROXIMATION THEORY 2, 233–246 (1969)

A Metric Condition for a Closed Circular Set to be a Set of Uniqueness

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A closed set on the circle (real line modulo 2π) is a set of uniqueness (U-set) if no nontrivial trigonometric series $\sum_0^\infty (a_n \cos nx + b_n \sin nx)$ converges to zero outside of the set. An alternative definition is the following: E is a U-set if we have $\overline{\lim}_{|n|\to\infty} |\hat{T}(n)| > 0$ for all pseudomeasures $T \neq 0$ carried by E (a pseudomeasure is a distribution T whose Fourier coefficients $\hat{T}(n)$ are bounded). We shall say that E is a *strong U-set* if we have $\overline{\lim}_{|n|\to\infty} |\hat{T}(n)| = \sup_n |\hat{T}(n)|$ for all pseudomeasures T carried by E.

A closed set on the circle is a set of multiplicity (M-set) if it is not a set of uniqueness. It is a set of multiplicity in the restricted sense (M_0-set) if it carries a measure $d\mu \neq 0$ whose Fourier coefficients $\hat{\mu}(n)$ tend to zero at infinity ([*3*], [*5*]).

It was proved by Ivašev–Musatov that no metric condition of a Hausdorff type implies that a set is a U-set. Namely, given any positive function $\phi(h)$ tending to 0 when $h \to 0$ $(h > 0)$, there exists an M_0 set whose Hausdorff measure with respect to the determining function $\phi(h)$ vanishes [*1*].

We shall consider a metric condition of another type. Given a closed circular set E, we write N_ϵ for the smallest number of closed intervals of length ϵ whose union contains E.

THEOREM 1. *If* $\underline{\lim}_{\epsilon\to 0} N\epsilon/\log 1/\epsilon = 0$, *E is a strong U-set.*

It is convenient to decompose the proof into two parts, and to introduce the notion of a Dirichlet set. A closed set E on the circle will be called a *Dirichlet set* if $\underline{\lim}_{n\to\infty} \sup_{x\in E} |\sin nx| = 0$ (the classical Dirichlet theorem on diophantine approximation implies that each finite set is a Dirichlet set).

1. *Under the assumption of Theorem* 1, *E is a Dirichlet set.*

The proof (not the statement) can be found in [*3*], p. 95; it is due to Salem

2. *A Dirichlet set E is a strong U-set.*

Here we use an idea which is familiar in this field ([*5*], p. 345; [*4*]). Let I_ϵ be the odd function defined by

$$\begin{aligned} I_\epsilon(x) &= x && \text{when } |x| \leqslant \epsilon \\ I_\epsilon(x) &= 2\epsilon - x && \text{when } \epsilon \leqslant x \leqslant 2\epsilon \\ I_\epsilon(x) &= 0 && \text{when } x \geqslant 2\epsilon. \end{aligned}$$

There exists an integral-valued function n_ϵ, tending to ∞ when $\epsilon \to 0$, such that $\sup_{x \in E} |\sin n_\epsilon x| < \epsilon$. For each pseudomeasure $T \neq 0$ carried by E, we have

$$\hat{T}(2n_\epsilon) - \hat{T}(0) = -2i\langle T, e^{-in_\epsilon x} I_\epsilon(\sin n_\epsilon x)\rangle$$

where $\langle T, f\rangle$ is written for $\sum_{n=-\infty}^{\infty} \hat{T}(n)\hat{f}(-n)$.

In the Banach space A of all functions f such that $\|f\|_A = \sum_{-\infty}^{\infty} |\hat{f}(n)| < \infty$, $I_\epsilon(\sin x)$ tends to zero as $\epsilon \to 0$. The same is true for $e^{in_\epsilon x} I_\epsilon(\sin n_\epsilon x)$. Therefore

$$\varliminf_{n\to\infty} |\hat{T}(n) - \hat{T}(0)| = 0.$$

In the same way

$$\varliminf_{n\to\infty} |\hat{T}(n) - \hat{T}(p)| = 0$$

for each given p; therefore $\varlimsup_{n\to\infty} |\hat{T}(n)| = \sup_p |\hat{T}(p)|$.

The condition given in Theorem 1 is the best possible, for we have the following result in the opposite direction.

THEOREM 2. *Given $\delta > 0$, there exists a closed circular set E such that* (1^0) *it carries a positive measure $d\mu$ of total mass* 1 *for which* $\varlimsup_{|n|\to\infty} |\hat{\mu}(n)| \leqslant \delta$, *and* ($2^0$) $N_\epsilon = O(\log(1/\epsilon))$ ($\epsilon \to 0$).

The proof is a slight refinement of that of Theorem 2 in [*2*]. The idea is to define E as a random set and $d\mu$ as a random measure, and to prove that the conclusion holds almost surely.

Given $\rho > 1$ (to be defined later), we write $r_j = \rho^{-2^j}$. Let F be the set of points

$$\sum_{j=1}^{\infty} \epsilon_j r_j \qquad (\epsilon_j = 0 \text{ or } 1)$$

and $d\sigma$ the measure

$$\tfrac{1}{2}(\delta_0 + \delta_{r_1}) * \tfrac{1}{2}(\delta_0 + \delta_{r_2}) * \ldots * \tfrac{1}{2}(\delta_0 + \delta_{r_j}) * \ldots;$$

F satisfies the condition $N_\epsilon = O(\log(1/\epsilon))$ and $d\sigma$ is the natural measure carried by F. We write $X(t) = X(\omega, t)$ for the random function of the circular brownian motion (that is, the Wiener function defined modulo 2π). We define E and $d\mu$ as the images of F and $d\sigma$ by $X(t)$. We denote by $\mathscr{E}(\ \ )$ the expectation of a random variable.

Since

$$\hat{\mu}(n) = \int e^{inx}\, d\mu(x) = \int e^{inX(t)}\, d\sigma(t),$$

a series of simple computations gives

$$\mathscr{E}(|\hat{\mu}(n)|^{2p}) \leqslant (p!)^2 \int_{t_1<t_2<\ldots<t_{2p}} \exp\left(-\frac{n^2}{2}(t_2 - t_1 + t_4 - t_3 + \ldots + t_{2p} - t_{2p-1})\right) \times d\sigma(t_1)\ldots d\sigma(t_{2p})$$

for each positive integer p [2]. Integrating first with respect to $t_2, t_4, \ldots t_{2p}$ and using the equalities

$$\sup_s \int e^{-(n^2/2)t}\, d\sigma(t+s) = \int e^{-(n^2/2)t}\, d\sigma(t) = \prod_{j=1}^{\infty} \tfrac{1}{2}(1 + e^{-(n^2/2)r_j}) = \psi(n),$$

$$\int_{t_1 < t_3 < \ldots < t_{2p-1}} d\sigma(t_1)\, d\sigma(t_3) \ldots d\sigma(t_{2p-1}) = \frac{1}{p!},$$

we obtain

$$\mathscr{E}(|\hat{\mu}(n)|^{2p}) \leqslant p!(\psi(n))^p < (p\psi(n))^p.$$

Given $\lambda > 0$, large, let us suppose

$$\lambda \rho^{2m} \leqslant n < \lambda \rho^{2m+1}.$$

We have

$$\tfrac{1}{2}(1 + e^{-(n^2/2)r_j}) \leqslant \tfrac{1}{2}\left(1 + \exp\left(-\frac{\lambda^2}{2}\rho^{2^{m+1}-2^j}\right)\right)$$

for $j = 1, 2, \ldots m+1$, and therefore

$$\psi(n) \leqslant \frac{C_{\lambda,\rho}}{2^{m+1}}$$

where $C_{\lambda,\rho}$ is near 1 when λ is large. From now on we suppose $C_{\lambda,\rho} < 2$. We choose $p = p(n) = h2^m$; h is a negative power of 2 and will be defined later; m is supposed to be large enough so that $h2^m \geqslant 1$. Then

$$\mathscr{E}(|\hat{\mu}(n)|^{2p}) \leqslant h^p$$

and

$$\mathscr{E}\left(\sum_{\lambda\rho^{2m} \leqslant n < \lambda\rho^{2m+1}} \left|\frac{\hat{\mu}(n)}{\delta}\right|^{2p}\right) \leqslant \lambda \rho^{2^{m+1}} (h\,\delta^{-2})^p$$

$$= \lambda(\rho^{2/h}\, h\, \delta^{-2})^p.$$

Given δ, we may define ρ and h in such a way that

$$\rho^{2/h}\, h\, \delta^{-2} < 1.$$

Then we have

$$\mathscr{E}\left(\sum_{n=1}^{\infty} \left|\frac{\hat{\mu}(n)}{\delta}\right|^{2p(n)}\right) < \infty;$$

therefore

$$\sum_{n=1}^{\infty} \left|\frac{\hat{\mu}(n)}{\delta}\right|^{2p(n)} < \infty \qquad \text{a. s.};$$

therefore $\overline{\lim}_{|n| \to \infty} |\hat{\mu}(n)| \leqslant \delta$ a. s.

For each $\epsilon > 0$, it is known that the function $X(t)$ satisfies a Lipschitz condition of order $\frac{1}{2} - \epsilon$. Since the set F satisfies $N_\epsilon = O(\log(1/\epsilon))$, it follows that E satisfies a. s. the same condition. This ends the proof of Theorem 2.

As a consequence of Theorem 1 and Theorem 2, *a necessary and sufficient condition on* $\phi(\epsilon)$ *that* $N_\epsilon = O(\phi(\epsilon))$ *implies that* E *is a strong U-set is* $\underline{\lim}_{\epsilon\to 0}\phi(\epsilon)/\log(1/\epsilon) = 0$. The same holds if we consider Dirichlet sets instead of strong U-sets.

The random set E constructed in the proof of Theorem 2 enjoys interesting properties; for example, it is a. s. independent over the rationals [*2*]. We were not able to prove that it is a. s. an M_0-set, nor to disprove it. It is easy to obtain a random M_0-set by changing the definition of r_j. We have

THEOREM 3. *Given any function* $A(\epsilon)$ *tending to* ∞ *when* $\epsilon \to 0$, *there exists an* M_0*-set* E *such that* $N_\epsilon = O(A(\epsilon)\log(1/\epsilon))$ $(\epsilon \to 0)$.

We leave the verification to the reader (similar statements can be found in [*2*]). Again, the set is a. s. independent.

As a consequence of Theorems 1 and 3, *a necessary and sufficient condition on* $\phi(\epsilon)$ *that* $N_\epsilon = o(\phi(\epsilon))$ *implies that* E *is a U-set is* $\underline{\lim}_{\epsilon\to 0}\phi(\epsilon)/\log(1/\epsilon) < \infty$.

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